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A symplectic formalism for quantum scattering on the hyperboloid

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Abstract. We study the quantum scattering of geodesic curves on the hyperboloid via established techniques in geometric quantisation. The wave operators are shown to be the dual Radon transform, and the S matrix thus calculated is the same as the one from Coulomb scattering. The procedure we used here can be extended to treat any non-compact symmetric spaces since they involve only geometric and group theoretic arguments.

1. Introduction

There has been an effort to study quantum scattering theory using dynamical groups of symmetries (Alhassid *et al* 1986). Specifically, the S matrix was essentially determined from purely group theoretical considerations in the presence of such symmetries. For the group SO(1, 3), the symmetry group of the Coulomb scattering, the S matrix was calculated (Wu 1985) from the eigenfunctions of the Laplace-Beltrami operator on the hyperboloid, on which the group acts naturally. The result agrees with the ones from Coulomb scattering even though the physical aspects of the problem never enter the calculation. These seem to suggest that the dynamical group plays a fundamental role in the scattering process, a viewpoint being used extensively in the bounded states case. Here we present a formal geometric procedure, using the SO(1, 3) group as our example, of getting the scattering wave operators and the S matrix. Our procedure involves creating a classical mechanical system having such a symmetry group and quantising that system using the techniques of geometric quantisation (Kostant 1970). Although our calculations are for the group SO(1, 3), it is clear that these ideas can be generalised to any semisimple Lie group of non-compact type with finite centre.

A natural model for a system with SO(1, 3) symmetry is the free motions on the hyperboloid, i.e. we take T^*H, the cotangent bundle over the hyperboloid, as our phase space with its standard symplectic structure, geodesic flow as the Hamiltonian vector field. We need to define the asymptotic data which are the subject of study in scattering theory. This is done in § 2.

We assume that readers are familiar with Kostant's geometric quantisation. (For a recent account see Sniatycki (1980).) We now discuss the quantum aspects of scattering. Newton (1980) has introduced a mixed representation space $L^2(S^2\times\mathbb{R}_+)$ for quantum scattering. The unitary transformation relating the standard position representation and this mixed representation is given formally by

$$\begin{aligned} &f(x)\in L^2(\mathbb{R}^3)\mapsto f(b,\lambda)\in L^2(S^2\times\mathbb{R}_+)\\ &f(b,\lambda)=\frac{\mathrm{d}}{\mathrm{d}\lambda}\int f(x)\delta(\langle x,b\rangle-\lambda)\,\mathrm{d}x \end{aligned}\tag{1.1}$$

i.e. a Radon transform (integrating over hyperplanes) followed by a derivative. The virtue of this representation is that the kinetic energy operator is simply $\mathrm{d}^2/\mathrm{d}\lambda^2$. This can be seen, symplectically, from the fact that if we extend b, λ to a canonical coordinate on $T^*\mathbb{R}^3$, letting $b = p/|p|$, the direction of the momentum p, the dual variable for λ is $|p|$. Thus quantising kinetic energy $|p|^2/2$ becomes differentiating with respect to λ. Physically λ measures the distance from the hyperplane to the scattering centre. This transformation captures information concerning asymptotic directions. The potential energy operator poses problems in this representation. However, since we are dealing with free motions, we can borrow the geometric ideas of this mixed representation. We will indeed show in § 3 that the Radon transform on the hyperboloid, which has been extensively studied (Helgason 1984), has the same physical significance as Newton's mixed representation, and thus can be viewed as an outgoing wave operator, in the sense that we can retrieve all asymptotic information. The incoming wave operator is defined analogously, via a change in orientation. From these we can calculate the scattering matrix. In § 4 we will remark briefly on the generalisation to other Lie groups.

2. The classical model

Let $H = \{(y_0, y_1, y_2, y_3) | [y, y] = y_0^2 - y_1^2 - y_2^2 - y_3^2 = 1, y_0 > 0\}$ be the unit hyperboloid in which the SO(1, 3) group acts naturally. We will henceforth denote by [,] the Lorentz inner product $(+---)$. The cotangent bundle T^*H has a standard symplectic structure. For convenience, we will think of T^*H as a reduction (Weinstein 1977) of the subset $\{(y, \eta) \in \mathbb{R}^4 \times \mathbb{R}^4 | [y, y] = 1\}$. Here $\mathbb{R}^4 \times \mathbb{R}^4$ has the symplectic form $\mathrm{d}\eta_0 \wedge \mathrm{d}y_0 - \Sigma\, \mathrm{d}\eta_i \wedge \mathrm{d}y_i$ so that the SO(1, 3) actions are canonical. We can identify $T^*H = \{(y, n) | [y, \eta] = 0, [y, y] = 1\}$.

It is known that H as a Riemannian manifold is isometric to the non-Euclidean ball. Thus the unit sphere S^2 is a natural boundary for H and points on the boundary are the asymptotic directions. It can be shown that the geodesics passing through y with outgoing direction $(t \to \infty)$, $k = (w_1, w_2, w_3) \in S^2$ can be parametrised as

$$c(t) = \begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{pmatrix} \cosh t - \left[\begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{pmatrix} - \frac{1}{[y, w]} \begin{pmatrix} 1 \\ w_1 \\ w_2 \\ w_3 \end{pmatrix} \right] \sinh t \tag{2.1}$$

where $w = (1, k)$.

In line with Newton's ideas, we need notion of hyperplanes and their distance from the scattering centre. The geometric significance of the hyperplanes is that each intersect a parallel family of lines normally. On H, these are known as horocycles (Helgason 1984) with defining equation $[y, w] = \text{constant}$, $w = (1, k)$, k is the direction of the parallel family. The distance between the horocycle and the scattering centre $(1, 0, 0, 0)$ is $-\log[y, w] = \lambda$. (Note that by convention the distance is negative if the horocycle has yet to propagate through the centre.) Since k is a unit vector, $\mathscr{E} = \mathrm{e}^{\lambda} w$ satisfies $[\mathscr{E}, \mathscr{E}] = 0$. Let $\Sigma = \{\mathscr{E} \in \mathbb{R}^4 | [\mathscr{E}, \mathscr{E}] = 0, \mathscr{E}_0 > 0\}$ on which SO(1, 3) acts. Σ parametrises the set of all horocycles $\mathscr{E} \to [y, \mathscr{E}] = 1$.

Let $(y, \eta) \in T^*H$ with outgoing direction k and distance λ from the centre. Define

$$\begin{aligned} &\chi_+ : T^*H \to \Sigma \\ &(y, \eta) \to \mathscr{E} = \mathrm{e}^{\lambda}(1, k). \end{aligned} \tag{2.2}$$

It can be shown that $\mathscr{E}$ is given explicitly as

$$\mathscr{E} = y + \frac{1}{(-[\eta, \eta])^{1/2}} \eta \tag{2.3}$$

($\eta \in T_y^* H$ implies $[\eta, \eta] < 0$). Note that χ_+ is SO(1, 3) action preserving. The pre-image of a point in Σ is a Lagrangian submanifold in T^*H. Thus χ_+ defines a polarisation of T^*H and we can quantise the space with respect to this polarisation. We denote the resulting Hilbert space as $Q(\chi_+)$.

The calculation is completely analogous in the incoming direction. We will denote the resulting Hilbert space by $Q(\chi_-)$ and will comment on it in § 4.

The standard position representation space is of course $L^2(H)$, square integrable functions with respect to the group invariant measure on H. Via the Blattner–Kostant–Sternberg (BKS) pairing (Blattner 1973), we can identify both $Q(\chi_-)$ and $Q(\chi_+)$ with $L^2(H)$. Thus we get the following diagram

$$\begin{tikzcd} Q(\chi_+) \arrow[rr, "S"] \arrow[dr, "\text{BKS}_+"'] & & Q(\chi_-) \arrow[dl, "\text{BKS}"] \\ & L^2(H) & \end{tikzcd}$$

where S will be the scattering matrix and the BKS will be the incoming and outgoing wave operators. Our objective is to show that $Q(\chi_+)$ is canonically isomorphic to $L^2(\Sigma)$ and the BKS_+ is the dual of the Radon transform of H.

3. Quantisation

Let $\alpha = \eta_0 \, dy_0 - \Sigma \eta_i \, dy_i$ be the connection form on T^*H considered as a subset of $\mathbb{R}^4 \times \mathbb{R}^4$ for convenience. $\omega = d\alpha$ is the symplectic form. $Q(\chi_+)$ consist of functions on T^*H covariant constant along the leaves of the polarisation with respect to α: if the leaves are defined by $f_k =$ constant, $k = 1, 2, 3$, then $\varphi \in Q(\chi_+)$ if

$$\{\mathscr{H}_{f_k} + i\langle \alpha, \mathscr{H}_{f_k} \rangle\} \varphi = 0 \quad \forall k. \tag{3.1}$$

Here

$$\mathscr{H}_f = (\partial f / \partial \eta_0) \, \partial / \partial y_0 - \sum (\partial f / \partial \eta_i) \, \partial / \partial y_i - (\partial f / \partial y_0) \, \partial / \partial \eta_0 + \sum (\partial f / \partial y_i) \, \partial / \partial \eta_i \tag{3.2}$$

is the Hamiltonian vector field of the function f with respect to ω. To ensure $\mathscr{H}_f$ is indeed a vector field on T^*H, we must choose f constant along the integral curves of $\mathscr{H}_{[y,y]}$ and $\mathscr{H}_{[y,\eta]}$. Then

$$\langle \alpha, \mathscr{H}_f \rangle = \eta_0 \, \partial f / \partial \eta_0 + \sum \eta_i \, \partial f / \partial \eta_i. \tag{3.3}$$

From (2.3), the defining functions are

$$y_i + \frac{1}{(-[\eta, \eta])^{1/2}} \eta_i = \text{constant}.$$

It can be shown that

$$\begin{aligned} f_i(y, \eta) &= \frac{1}{([y, y])^{1/2}} \left(y_i + \frac{y_i[y, \eta] + [y, y]\eta_i}{([y, \eta]^2 - [\eta, \eta][y, y])^{1/2}} \right) \\ &= y_i + \frac{1}{(-[\eta, \eta])^{1/2}} \eta_i \quad \text{on } T^*H \end{aligned} \tag{3.4}$$

and these are constant along $\mathscr{H}_{[y,y]}$ and $\mathscr{H}_{[y,\eta]}$. Thus the polarisation is defined by the Hamiltonian vector fields of f_k. Furthermore, $f_k(y, \eta)$ are homogeneous of degree 0 in the variables η. So $\langle \alpha, \mathscr{H}_{fk}\rangle = 0$ and the covariant constant condition becomes constant along the leaves, i.e. the pull back

$$\chi_+^*: L^2(\Sigma) \to Q(\chi_+) \tag{3.5}$$

is a canonical isomorphism. Here the symbol L^2 is used loosely, since we are only interested in the geometrical contents of the integral formulae rather their convergence. The Hilbert space structures are dictated by the group actions.

The position representation space arises from quantising T^*H with respect to the vertical polarisation whose leaves are the cotangent planes T_y^*H. Since the vector field

$$v = \eta_0\, \partial/\partial\eta_0 + \sum \eta_i\, \partial/\partial\eta_i \tag{3.6}$$

is tangent to the leaves of both polarisations, these polarisations do not intersect transversely. The BKS pairing technique has to be modified (Sniatycki 1980). Roughly the pairing involves integrating not on T^*H, but on a subset which cuts each integral curve of v once. For convenience, we choose the subset $[\eta, \eta] = -1$. Then

$$\langle \varphi(y, \eta), \psi(y)\rangle_{\mathrm{BKS}} = \int_H \mathrm{d}y\, \psi^*(y) \int_{T_y^*H \cap [\eta,\eta]=-1} \varphi(y, \eta)(\det \mathrm{d}\eta)^{1/2}. \tag{3.7}$$

Here det is calculated as follows: let $\{v, v_1, v_2\}$ and $\{v, w_1, w_2\}$ be vector fields for the two polarisations in question and v as in (3.6). Then det is the determinant of the 2×2 matrix whose ijth entry is $\omega(v_i, w_j)$. Each entry is a constant for the following reason: ω is SO(1, 3) invariant, since the group action is symplectic. The vector field can be chosen SO(1, 3) equivariantly since the leaves, restricted to the subset we are integrating over, are orbits of subgroups of SO(1, 3), $T_y^*H \cap [\eta, \eta] = -1$ is the orbit of the isotropy subgroup of y, $\chi_+^{-1}(\mathscr{E}) \cap [\eta, \eta] = -1$ is the orbit of the isotropy subgroup of $\mathscr{E}$, so $\omega(v_i, w_j)$ is an SO(1, 3) invariant function. The subset $[\eta, \eta] = -1$ is an SO(1, 3) orbit now implies that it is a constant. Thus

$$\begin{aligned}\langle \varphi, \psi\rangle_{\mathrm{BKS}} &= \int_H \mathrm{d}y\, \psi^*(y) \int_{T_y^*H \cap [\eta,\eta]=-1} \varphi(y, \eta)\, \mathrm{d}\eta \\ &= \int_H \mathrm{d}y\, \psi^*(y) \int_{[y,\mathscr{E}]=1} \varphi(\mathscr{E})\, \mathrm{d}\mathscr{E}.\end{aligned} \tag{3.8}$$

BKS induces an isomorphism $L^2(\Sigma) \to L^2(H)$

$$\Phi(y) = \int_{[y,\mathscr{E}]=1} \varphi(\mathscr{E})\, \mathrm{d}\mathscr{E} \tag{3.9}$$

i.e. integrating over the set of horocycles containing y. This is indeed the dual Radon transform (Helgason 1984).

The eigensubspaces of the energy operator should be the irreducible subspaces of the SO(1, 3) representation. In the spirit of Alhassid *et al*, there is no need to define such an operator. Of course one can easily show that the Laplace operator on $L^2(H)$ (Casimir operator) is the quantisation of the energy $(-[\eta, \eta])^{1/2}$.

The relation between the Radon transform and dual Radon transform is

$$\Box R^* R\varphi(y) = c\varphi(y) \tag{3.10}$$

where R, R^* and $\Box$ denote the Radon, its dual and the Laplace-Beltrami operator respectively. c is a constant depending only on the Lie group and not even on the eigensubspace. Since we are only interested in projective representations in quantum mechanics, we can indeed interpret R as the inverse wave operator.

The irreducible subspaces in $L^2(\Sigma)$ are homogeneous functions in $\mathscr{E}_0$ of degree $-\mathrm{i}\rho-1$, ρ real (Gelfand *et al* 1966). The function

$$\varphi(\mathscr{E}_0, \mathscr{E}_0\theta) = \mathscr{E}_0^{-\mathrm{i}\rho-1}\delta_k(\theta) \tag{3.11}$$

represents a case of interest,

$$\Phi(y) = [y, k]^{\mathrm{i}\rho-1}. \tag{3.12}$$

These are the plane waves with outgoing direction k and energy ρ. One can show that the plane wave with incoming direction k' and energy ρ is $[y, k']^{-\mathrm{i}\rho-1}$. This is in fact the motivation behind the calculation of the S matrix in Wu (1985):

$$S_\rho(k, k') = \int [y, k]^{\mathrm{i}\rho-1}[y, k']^{\mathrm{i}\rho-1}\,\mathrm{d}y. \tag{3.13}$$

This result is consistent with Coulomb scattering.

The group theoretical significance of the S matrix is seen in the context of the Lie group and their Iwasawa decompositions.

4. Generalisations

Let G be a semisimple Lie group of non-compact type with finite centre, K a maximal compact subgroup. Fix a Cartan decomposition $\boldsymbol{p}\oplus\boldsymbol{k}$ of the Lie algebra, $\boldsymbol{a}$ a maximal Abelian subalgebra in $\boldsymbol{p}$, $\boldsymbol{a}^+$ a Weyl chamber, set $A = \exp \boldsymbol{a}$, let M, M' denote the centraliser and normaliser of A in K, then K/M can be viewed as the boundary (asymptotic directions) of the symmetric space $X = \mathrm{G}/\mathrm{K}$ (Helgason 1984). Let $\mathrm{G} = \mathrm{K}A\mathrm{N}$ be the Iwasawa decomposition corresponding to our previous choices. It is important to point out that N, the nilpotent subgroup, depends on the choice of the Weyl chamber $\boldsymbol{a}^+$. The set of horocycles in X is parametrised by $\Sigma = \mathrm{G}/M\mathrm{N} = \mathrm{K}/M \times A$. G also acts on the boundary via $\mathrm{K}/M = \mathrm{G}/MA\mathrm{N}$.

Modelled on the SO(1, 3) case, we consider geodesic motions on X and can define a G equivariant map

$$\chi_+ : T^*X \to \Sigma \tag{4.1}$$

where $\chi_+(x, \eta) = \mathscr{E} = b \exp \lambda M\mathrm{N}$, bM specifies the asymptotic outgoing direction, $\lambda \in \boldsymbol{a}$ specifies the 'complex distance' between the horocycle $\mathscr{E}$ and the scattering centre $e\mathrm{K} \in \mathrm{G}/\mathrm{K}$. Explicitly, if we identify the cotangent bundle via the Killing form, let $x = h\mathrm{K}$ where $h = k \exp ak^{-1}$, we can parallel translate $h_*^{-1}\eta \in T_eX \simeq \boldsymbol{p}$. Thus $h_*^{-1}\eta = \kappa\alpha\kappa^{-1}$ where $\kappa \in \mathrm{K}/M$, $\alpha \in \boldsymbol{a}^+$ are uniquely defined. Then

$$b = h\kappa MA\mathrm{N} \tag{4.2}$$

and λ is such that there is a $n \in \mathrm{N}$ with

$$x = b(\exp \lambda)n\mathrm{K}. \tag{4.3}$$

Since λ is a vector quantity, its dual variable, the 'complex energy', should be the same, i.e. we have a set of dim $\boldsymbol{a}$ conserved quantities in the dynamical system. We have an interesting physical interpretation of the Iwasawa decomposition: consider the G action on X, K acts on the set of asymptotic directions, the orbits of conjugate subgroups of A are the families of parallel trajectories and those of N are the horocycles, integrating over which gives the wave operator.

In the incoming direction, since there is a change in orientation, we should decompose $\boldsymbol{p}$ as $\mathrm{K}/M \times \boldsymbol{a}^-$ ($\boldsymbol{a}^- = -\boldsymbol{a}^+$), or the expressions (4.1)-(4.3) are exactly the same except that the nilpotent subgroup N arises from choosing $\boldsymbol{a}^-$ as the Weyl chamber in the group decomposition. For the SO(1, 3) group, $\boldsymbol{a}$ is one dimensional, so there are only two different Weyl chambers. Our work suggest that the S matrix relates the Iwasawa decompositions corresponding to the two choices, which we conveniently labelled as incoming and outgoing. It will be of interest to see whether the presence of more chambers introduces other internal symmetries in a higher-dimensional case. In particular, the role of the Weyl group, M'/M, which permutes them. The SO(3, 2) group deserves special attention since it has to do with heavy-ion scattering.

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